

On the parameterized complexity of cutting a few vertices from a graph¹

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Abstract. We study the parameterized complexity of separating a small set of vertices from a graph by a small vertex-separator. That is, given a graph G and integers k, t , the task is to find a vertex set X with $|X| \leq k$ and $|N(X)| \leq t$. We show that

- the problem is fixed-parameter tractable (FPT) when parameterized by t but W[1]-hard when parameterized by k , and
- a terminal variant of the problem, where X must contain a given vertex s , is W[1]-hard when parameterized either by k or by t alone, but is FPT when parameterized by $k + t$.

We also show that if we consider edge cuts instead of vertex cuts, the terminal variant is NP-hard and FPT parameterized by $k + t$.

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1 Introduction

We investigate two related problems that concern separating a small vertex set from a graph $G = (V, E)$. Specifically, we consider finding a vertex set X of size at most k such that

1. X is separated from the rest of V by a small cut (e.g. *finding communities in a social network*, cf. [13]), or
2. X is separated from the rest of V by a small cut and contains a specified terminal vertex s (e.g. *isolating a dangerous node*, cf. [10, 12]).

We focus on *parameterized complexity* of the *vertex-cut versions* of these problems.

Parameterized vertex cuts. Our interest in the vertex-cut version stems from the following parameterized separation problem, studied by Marx [14]. Let $N(X)$ denote the vertex-neighborhood of X .

CUTTING k VERTICES

Input: Graph $G = (V, E)$, integers $k \geq 1, t \geq 0$

Parameter 1: k

Parameter 2: t

Question: Is there a set $X \subseteq V$ such that $|X| = k$ and $|N(X)| \leq t$?

In particular, Marx showed that CUTTING k VERTICES is $W[1]$ -hard even when parameterized by both k and t . We contrast this result by investigating the parameterized complexity of the two related separation problems with relaxed requirement on the size of the separated set X .

CUTTING AT MOST k VERTICES

Input: Graph $G = (V, E)$, integers $k \geq 1, t \geq 0$

Parameter 1: k

Parameter 2: t

Question: Is there a non-empty set $X \subseteq V$ such that $|X| \leq k$ and $|N(X)| \leq t$?

CUTTING AT MOST k VERTICES WITH TERMINAL

Input: Graph $G = (V, E)$, terminal vertex s , integers $k \geq 1, t \geq 0$

Parameter 1: k

Parameter 2: t

Question: Is there a non-empty set $X \subseteq V$ such that $s \in X$, $|X| \leq k$ and $|N(X)| \leq t$?

We show that these closely related problems exhibit quite different complexity behaviors. In particular, we show that CUTTING AT MOST k VERTICES is fixed-parameter tractable (FPT) when parameterized by the size of the separator t , while we need both k and t as parameters to obtain an FPT algorithm for CUTTING AT MOST k VERTICES WITH TERMINAL. A full summary of the parameterized complexity of these problems and our results is given in Table 1.

The main algorithmic contribution of our paper is the proof that CUTTING AT MOST k VERTICES is FPT when parameterized by t (Theorem 5). To obtain

this result, we utilize the concept of *important separators* introduced by Marx [14]. However, a direct application of important separators—guess a vertex contained in the separated set, and find a minimal set containing this vertex that can be separated from the remaining graph by at most t vertices—does not work. Indeed, pursuing this approach would bring us to essentially solving CUTTING AT MOST k VERTICES WITH TERMINAL, which is W[1]-hard when parameterized by t . Our FPT algorithm is based on new structural results about unique important separators of minimum size separating pairs of vertices. We also observe that it is unlikely that CUTTING AT MOST k VERTICES has a polynomial kernel.

Edge cuts. Although our main focus is on vertex cuts, we will also make some remarks on the edge-cut versions of the problems. In particular, the edge-cut versions again exhibit a different kind of complexity behavior. Let $\partial(X)$ denote the edge-boundary of X .

CUTTING AT MOST k VERTICES BY EDGE-CUT

Input: Graph $G = (V, E)$, integers $k \geq 1, t \geq 0$

Parameter 1: k

Parameter 2: t

Question: Is there a non-empty set $X \subseteq V$ such that $|X| \leq k$ and $|\partial(X)| \leq t$?

CUTTING k VERTICES BY EDGE-CUT WITH TERMINAL

Input: Graph $G = (V, E)$, terminal vertex s , integers $k \geq 1, t \geq 0$

Parameter 1: k

Parameter 2: t

Question: Is there a set $X \subseteq V$ such that $s \in X$, $|X| \leq k$ and $|\partial(X)| \leq t$?

Results by Watanabe and Nakamura [17] imply that CUTTING AT MOST k VERTICES BY EDGE-CUT can be done in polynomial time even when k and t are part of the input; more recently, Armon and Zwick [2] have shown that this also holds in the edge-weighted case. However, we show that CUTTING AT MOST k VERTICES BY EDGE-CUT WITH TERMINAL is NP-hard (Theorem 14) and FPT when parameterized by k and t (Theorem 4). It would also be interesting to know whether the problem is FPT when parameterized by k or t only; we leave this as an open question.

Related edge-cut problems have received attention in the context of approximation algorithms. In contrast to CUTTING AT MOST k VERTICES BY EDGE-CUT,

Parameter	CUTTING k VERTICES	CUTTING $\leq k$ VERTICES	CUTTING $\leq k$ VERTICES WITH TERMINAL
k	W[1]-hard, [14]	W[1]-hard, Thm 11	W[1]-hard, Thm 11
t	W[1]-hard, [14]	FPT, Thm 5	W[1]-hard, Thm 13
k and t	W[1]-hard, [14]	FPT, Thm 4	FPT, Thm 4

Table 1: Parameterized complexity of CUTTING k VERTICES, CUTTING AT MOST k VERTICES, and CUTTING AT MOST k VERTICES WITH TERMINAL.

finding a minimum-weight edge-cut that separates exactly k vertices is NP-hard. Feige et al. [8] give a PTAS for $k = O(\log n)$ and an $O(k/\log n)$ -approximation for $k = \Omega(\log n)$; Li and Zhang [13] give an $O(\log n)$ -approximation. Approximation algorithms have also been given for unbalanced s - t -cuts, where s and t are specified terminal vertices and the task is to find an edge cut $(X, V \setminus X)$ with $s \in X$ and $t \in V \setminus X$ such that (a) $|X| \leq k$ and weight of the cut is minimized [10, 13], or (b) weight of the cut is at most w and $|X|$ is minimized [12].

2 Basic definitions and preliminaries

Graph theory. We follow the conventions of Diestel [6] with graph-theoretic notations. We only consider finite, undirected graphs that do not contain loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$, or simply by V and E , respectively. Typically we use n to denote the number of vertices of G and m the number of edges.

For a set of vertices $U \subseteq V(G)$, we write $G[U]$ for the subgraph of G induced by U , and $G - U$ for the graph obtained from G by the removal of all the vertices of U , i.e., the subgraph of G induced by $V(G) \setminus U$. Similarly, for a set of edges A , the graph obtained from G by the removal of all the edges in A is denoted by $G - A$.

For a vertex v , we denote by $N_G(v)$ its (*open*) *neighborhood*, that is, the set of vertices which are adjacent to v . The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. For a set of vertices $U \subseteq V(G)$, we write $N_G(U) = \cup_{v \in U} N_G(v) \setminus U$ and $\partial_G(U) = \{uv \in E(G) \mid u \in U, v \in V(G) \setminus U\}$. We may omit subscripts in these notations if there is no danger of ambiguity.

Submodularity. We will make use of the well-known fact that given a graph G , the mapping $2^V \rightarrow \mathbb{Z}$ defined by $U \mapsto |N(U)|$ is *submodular*. That is, for $A, B \subseteq V$ we have

$$|N(A \cap B)| + |N(A \cup B)| \leq |N(A)| + |N(B)|. \quad (1)$$

Important separators. Let G be a graph. For disjoint sets $X, Y \subseteq V$, a vertex set $S \subseteq V \setminus (X \cup Y)$ is a (*vertex*) (X, Y) -*separator* if there is no path from X to Y in $G - S$. An *edge* (X, Y) -*separator* $A \subseteq E$ is defined analogously. Note that we do not allow deletion of vertices in X and Y , and thus there are no vertex (X, Y) -separators if X and Y are adjacent. As our main focus is on the vertex-cut problems, all separators are henceforth vertex-separators unless otherwise specified.

We will make use of the concept of *important separators*, introduced by Marx [14]. A vertex v is *reachable* from a set $X \subseteq V$ if G has a path that joins a vertex of X and v . For any sets S and $X \subseteq V \setminus S$, we denote the set of vertices reachable from X in $G - S$ by $R(X, S)$. An (X, Y) -separator S is *minimal* if no proper subset of S is an (X, Y) -separator. For (X, Y) -separators S and T , we say that T *dominates* S if $|T| \leq |S|$ and $R(X, S)$ is a proper subset of $R(X, T)$. For singleton sets, we will write x instead of $\{x\}$ in the notations defined above.

Definition 1 ([14]). An (X, Y) -separator S is *important* if it is minimal and there is no other (X, Y) -separator dominating S .

In particular, this definition implies that for any (X, Y) -separator S there exists an important (X, Y) -separator T with $|T| \leq |S|$ and $R(X, T) \supseteq R(X, S)$. If S is not important, then at least one of the aforementioned relations is proper.

The algorithmic usefulness of important separators follows from the fact that the number of important separators of size at most t is bounded by t alone, and furthermore, these separators can be listed efficiently. Moreover, minimum-size important separators are unique and can be found in polynomial time. That is, we will make use of the following lemmas.

Lemma 2 ([5]). *For any disjoint sets $X, Y \subseteq V$, the number of important (X, Y) -separators of size at most t is at most 4^t , and all important (X, Y) -separators of size at most t can be listed in time $4^t \cdot n^{O(1)}$.*

Lemma 3 ([14]). *For any sets $X, Y \subseteq V$, if there exists an (X, Y) -separator, then there is exactly one important (X, Y) -separator of minimum size. This separator can be found in polynomial time.*

Parameterized complexity. We will briefly review the basic notions of parameterized complexity, though we refer to the books of Downey and Fellows [7], Flum and Grohe [9], and Niedermeier [16] for a detailed introduction. Parameterized complexity is a two-dimensional framework for studying the computational complexity of a problem; one dimension is the input size n and another one is a parameter k . A parameterized problem is *fixed-parameter tractable* (or FPT) if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f , and in the class XP if it can be solved in time $O(n^{f(k)})$ for some function f .

Between FPT and XP lies the class W[1]. One of basic assumptions of the parameterized complexity theory is the conjecture that $W[1] \neq \text{FPT}$, and it is thus held to be unlikely that a W[1]-hard problem would be in FPT. For exact definition of W[1], we refer to the books mentioned above. We mention only that INDEPENDENT SET and CLIQUE parameterized by solution size are two fundamental problems that are known to be W[1]-complete.

The basic way of showing that a parameterized problem is unlikely to be fixed-parameter tractable is to prove W[1]-hardness. To show that a problem is W[1]-hard, it is enough to give a *parameterized reduction* from a known W[1]-hard problem. That is, let A, B be parameterized problems. We say that A is (uniformly many-one) FPT-reducible to B if there exist functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, a constant $c \in \mathbb{N}$ and an algorithm \mathcal{A} that transforms an instance (x, k) of A into an instance $(x', g(k))$ of B in time $f(k)|x|^c$ so that $(x, k) \in A$ if and only if $(x', g(k)) \in B$.

Cutting problems with parameters k and t . In the remainder of this section, we consider CUTTING AT MOST k VERTICES, CUTTING AT MOST k VERTICES WITH TERMINAL, and CUTTING AT MOST k VERTICES BY EDGE-CUT WITH TERMINAL with parameters k and t . We first note that if there exists a solution for one of the problems, then there is also a solution in which X is connected; indeed, it suffices to take any maximal connected $Y \subseteq X$. Furthermore, we note finding a connected set X with $|X| = k$ and $|N(X)| \leq t$ is fixed-parameter tractable with parameters k and t due to a result by Marx [14, Theorem 13], and thus CUTTING AT MOST k VERTICES is also fixed-parameter tractable with parameters k and t .

We now give a simple color-coding algorithm [1, 4] for the three problems with parameters k and t , in particular improving upon the running time of the aforementioned algorithm for CUTTING AT MOST k VERTICES.

Theorem 4. CUTTING AT MOST k VERTICES, CUTTING AT MOST k VERTICES WITH TERMINAL, and CUTTING AT MOST k VERTICES BY EDGE-CUT WITH TERMINAL can be solved in time $2^{k+t} \cdot (k+t)^{O(\log(k+t))} \cdot n^{O(1)}$.

Proof. We first consider a 2-colored version of CUTTING AT MOST k VERTICES. That is, we are given a graph G where each vertex is either colored red or blue (this is not required to be a proper coloring), and the task is to find a connected red set X with $|X| \leq k$ such that $N(X)$ is blue and $|N(X)| \leq t$. If such a set exists, it can be found in polynomial time by trying all maximal connected red sets.

Now let $G = (V, E)$ be a graph. Assume that there is a set X with $|X| \leq k$ and $|N(X)| \leq t$; we may assume that X is connected. It suffices to find a coloring of V such that X is colored red and $N(X)$ is colored blue. This can be done by coloring each vertex v either red or blue independently and uniformly at random. Indeed, this gives a desired coloring with probability at least $2^{-(k+t)}$, which immediately yields a $2^{k+t} \cdot n^{O(1)}$ time randomized algorithm for CUTTING AT MOST k VERTICES.

This algorithm can be derandomized in standard fashion using universal sets (compare with Cai et al. [4]). Recall that a (n, ℓ) -universal set is a collection of binary vectors of length n such that for each index subset of size ℓ , each of the 2^ℓ possible combinations of values appears in some vector of the set. A construction of Naor et al. [15] gives a (n, ℓ) -universal set of size $2^\ell \cdot \ell^{O(\log \ell)} \log n$ that can be listed in linear time. It suffices to try all colorings induced by a $(n, k+t)$ -universal set obtained through this construction.

The given algorithm works for CUTTING AT MOST k VERTICES WITH TERMINAL with obvious modifications. That is, given a coloring, we simply check if the terminal s is red and its connected red component is a solution. This also works for CUTTING AT MOST k VERTICES BY EDGE-CUT WITH TERMINAL, as we have $|N(X)| \leq |\partial(X)|$. \square

3 Cutting at most k vertices parameterized by t

In this section we show that CUTTING AT MOST k VERTICES is fixed-parameter tractable when parameterized by the size of the separator t only. Specifically, we will prove the following theorem.

Theorem 5. CUTTING AT MOST k VERTICES can be solved in time $4^t \cdot n^{O(1)}$.

The remainder of this section consists of the proof of Theorem 5. Note that we may assume $\frac{3}{4}t < k < n - t$. Indeed, if $k \leq ct$ for a fixed constant $c < 1$, then we can apply the algorithm of Theorem 4 to solve CUTTING AT MOST k VERTICES in time $4^t n^{O(1)}$. On the other hand, if $k \geq n - t$, then any vertex set X of size k is a solution, as $|N(X)| \leq n - k \leq t$.

We start by guessing a vertex $u \in V$ that belongs to a solution if one exists; specifically, we can try all choices of u . With u fixed, we compute for each $v \in V \setminus (\{u\} \cup N(u))$ the unique minimum important (u, v) -separator S_v . This can

be done in polynomial time by Lemma 3. Let V_0 be set of those v with $|S_v| \leq t$, and denote $R(v) = R(v, S_v)$. Finally, let \mathcal{X} be a set family consisting of those $R(v)$ for $v \in V_0$ that are inclusion-minimal, i.e., if $R(v) \in \mathcal{X}$, then there is no $w \in V_0$ such that $R(w) \subsetneq R(v)$. Note that we can compute the sets V_0 , $R(v)$ and \mathcal{X} in polynomial time.

There are now three possible cases that may occur.

1. If $V_0 = \emptyset$, we conclude that we have no solution containing u .
2. If there is $v \in V_0$ such that $|R(v)| \leq k$, then $X = R(v)$ gives a solution, and we stop and return a YES-answer.
3. Otherwise, \mathcal{X} is non-empty and for all sets $A \in \mathcal{X}$ we have $|A| > k$.

We only have to consider the last case, as otherwise we are done. We will show that in that case, the sets $A \in \mathcal{X}$ can be used to find a solution X containing u if one exists. For this, we need the following structural results about the sets $R(v)$.

Lemma 6. *For any $v, w \in V_0$, if $w \in R(v)$ then $R(w) \subseteq R(v)$.*

Proof. Let $A = R(v)$ and $B = R(w)$. Since $S_v = N(A)$ is a (u, v) -separator of minimum size, we must have $|N(A \cup B)| \geq |N(A)|$. By (1), we have

$$|N(A \cap B)| \leq |N(A)| + |N(B)| - |N(A \cup B)| \leq |N(B)|.$$

Because $w \in A$, the set $N(A \cap B)$ is a (u, w) -separator. Thus, if $B \neq A \cap B$, then $N(A \cap B)$ is a (u, w) -separator that witnesses that S_w is not an important separator. But this is not possible by the definition of S_w , so we have $B = A \cap B \subseteq A$. \square

Lemma 7. *Any distinct $A, B \in \mathcal{X}$ are disjoint.*

Proof. Assume that $A, B \in \mathcal{X}$ are distinct and intersect. Then there is $v \in A \cap B$. Since $v \in A$, the set $N(A)$ is a (u, v) -separator of size at most t , and $v \in V_0$. Recall that \mathcal{X} contains inclusion-minimal sets $R(w)$ for $w \in V_0$. But by Lemma 6, $R(v)$ is a proper subset of both A and B , which is not possible by the definition of \mathcal{X} . \square

Now assume that the input graph G has a solution for CUTTING AT MOST k VERTICES containing u . In particular, then there is an inclusion-minimal set $X \subseteq V$ with $u \in X$ satisfying $|X| \leq k$ and $|N(X)| \leq t$. Let us fix one such set X .

Lemma 8. *For all $A \in \mathcal{X}$, the set A is either contained in $X \cup N(X)$ or does not intersect it.*

Proof. Suppose that there is a set $A \in \mathcal{X}$ that intersects both $X \cup N(X)$ and its complement. Let $Y = V \setminus (X \cup N(X))$.

Now let $v \in A \cap Y$. By Lemma 6, we have $R(v) = A$. If $|N(A \cup Y)| > |N(Y)|$ then it follows from (1) that

$$|N(A \cap Y)| \leq |N(A)| + |N(Y)| - |N(A \cup Y)| < |N(A)|.$$

However, this would imply that $N(A \cap Y)$ is a (u, v) -separator smaller than $S_v = N(A)$.

Thus, we have $|N(A \cup Y)| \leq |N(Y)|$. But $X' = X \setminus (A \cup Y \cup N(A \cup Y))$ is a proper subset of X ; furthermore, any vertex of $N(X')$ that is not in $N(A \cup Y)$ is also in $N(X) \setminus N(Y)$, so we have $|N(X')| \leq |N(X)| \leq t$. This is in contradiction with the minimality of X . \square

Lemma 9. *Let Z be the union of all $A \in \mathcal{X}$ that do not intersect $X \cup N(X)$. Then $Z \neq \emptyset$ and there is an important (Z, u) -separator S of size at most t such that $|R(u, S)| + |S| \leq k + t$.*

Proof. Let $S = N(X)$. Consider an arbitrary $v \in V \setminus (X \cup S)$; such vertex exists, since $k + t < n$. Since S separates v from u , the set $R(v)$ is well-defined.

Suppose now that $R(v)$ is not contained in $R(v, S)$. Let $B = R(u, S_v)$. Since S_v is a minimum-size (u, v) -separator we have $|N(B)| = |N(R(v))|$. But $N(X \cup B)$ also separates u and v , so we have $|N(X \cup B)| \geq |N(R(v))| = |N(B)|$. By (1), we have

$$|N(X \cap B)| \leq |N(X)| + |N(B)| - |N(X \cup B)| \leq |N(X)| \leq t.$$

But since $R(v)$ is not contained in $R(v, S)$, it follows that $X \cap B$ is a proper subset of X , which contradicts the minimality of X .

Thus we have $R(v) \subseteq R(v, S)$. It follows that $R(v, S)$ contains a set $A \in \mathcal{X}$, which implies that $Z \neq \emptyset$ and $v \in R(A, S) \subseteq R(Z, S)$. Furthermore, since $v \in V \setminus (X \cup S)$ was chosen arbitrarily, we have that $R(Z, S) = V \setminus (X \cup S)$.

If S is an important (Z, u) -separator, we are done. Otherwise, there is an important (Z, u) -separator T with $|T| \leq |S|$ and $R(Z, S) \subseteq R(Z, T)$. But then we have $|T| \leq t$, and $R(u, T) \cup T \subseteq X \cup S$, that is, $|R(u, T) \cup T| \leq k + t$. \square

Recall now that we may assume $|A| > k$ for all $A \in \mathcal{X}$. Furthermore, we have $|X \cup N(X)| \leq k + t < \left(2 + \frac{1}{3}\right)k$ and the sets $A \in \mathcal{X}$ are disjoint by Lemma 7. Thus, at most two sets $A \in \mathcal{X}$ fit inside $X \cup N(X)$ by Lemma 8. This means that if we let Z be the union of all $A \in \mathcal{X}$ that do not intersect $X \cup N(X)$, then as we have already computed \mathcal{X} , we can guess Z by trying all $O(n^2)$ possible choices.

Assume now that X is a minimal solution containing u and our guess for Z is correct. We enumerate all important (Z, u) -separators of size at most t . We will find by Lemma 9 an important (Z, u) -separator S such that $|S| \leq t$ and $|R(u, S)| + |S| \leq k + t$. If $|R(u, S)| \leq k$, we have found a solution. Otherwise, we delete a set S' of $|R(u, S)| - k$ elements from $R(u, S)$ to obtain a solution X' . To see that this suffices, observe that $N(X') \subseteq S' \cup S$. Therefore, $|N(X')| \leq |S'| + |S| = |R(u, S)| - k + |S| \leq t$. As all important (Z, u) -separators can be listed in time $4^t \cdot n^{O(1)}$ by Lemma 2, the proof of Theorem 5 is complete.

Remark 10. While finding one set $X \subseteq V$ such that $|X| \leq k$ and $|N(X)| \leq t$ is FPT with the parameter t , we need time $n^{O(t)}$ to list all such inclusion-minimal sets. Indeed, suppose that G is a complete graph and $t = n - k$. Then for any of the $\binom{n}{t} = \binom{n}{k}$ subsets X of size k , we have $|N(X)| \leq t$, and X is an inclusion-minimal set with this property.

4 Hardness results

We start this section by complementing Theorem 4, as we show that CUTTING AT MOST k VERTICES is NP-complete and W[1]-hard when parameterized by k . We also show that same holds for CUTTING AT MOST k VERTICES WITH TERMINAL. Note that both of these problems are in XP when parameterized by k , as they can be solved by checking all vertex subsets of size at most k .

Theorem 11. *CUTTING AT MOST k VERTICES and CUTTING AT MOST k VERTICES WITH TERMINAL are NP-complete and W[1]-hard with the parameter k .*

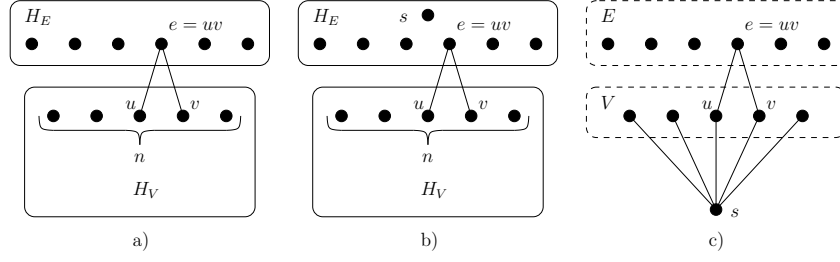


Figure 1: Constructions of G' in the proofs of Theorems 11 and 13.

Proof. We prove the $W[1]$ -hardness claim for CUTTING AT MOST k VERTICES by a reduction from CLIQUE. Recall that this $W[1]$ -complete (see [7]) problem asks for a graph G and a positive integer k where k is a parameter, whether G contains a clique of size k . Let (G, k) be an instance of CLIQUE, $n = |V(G)|$ and $m = |E(G)|$; we construct an instance (G', k', t) of CUTTING AT MOST k VERTICES as follows. Let H_V be a clique of size n^3 and identify n vertices of H_V with the vertices of G . Let H_E be a clique of size m and identify the vertices of H_E with the edges of G . Finally, add an edge between vertex v of H_V and vertex e of H_E whenever v is incident to e in G . Set $k' = \binom{k}{2}$ and $t = k + m - \binom{k}{2}$. The construction is shown in Fig. 1 a).

If G has a k -clique K , then for the set X that consists of the vertices e of H_E corresponding to edges of K we have $|X| = \binom{k}{2}$ and $|N_{G'}(X)| = k + m - \binom{k}{2}$. On the other hand, suppose that there is a set of vertices X of G' such that $|X| \leq k'$ and $|N_{G'}(X)| \leq t$. First, we note that X cannot contain any vertices of H_V , as then $N_{G'}(X)$ would be too large. Thus, the set X consists of vertices of H_E . Furthermore, we have that $|X| = \binom{k}{2}$. Indeed, assume that this is not the case. If $|X| \leq \binom{k-1}{2} = \binom{k}{2} - k$, then, since X has at least one neighbor in H_V , we have

$$|N_{G'}(X)| \geq m - |X| + 1 \geq m - \binom{k}{2} + k + 1,$$

and if $\binom{k-1}{2} < |X| < \binom{k}{2}$, then X has at least k neighbors in H_V , and thus

$$|N_{G'}(X)| \geq m - |X| + k > m - \binom{k}{2} + k.$$

Thus, we have that X only consist of vertices of H_E and $|X| = \binom{k}{2}$. But then the vertices of H_V that are in $N_{G'}(X)$ form a k -clique in G .

The $W[1]$ -hardness proof for CUTTING AT MOST k VERTICES WITH TERMINAL uses the same arguments. The only difference is that we add the terminal s in the clique H_E and let $k' = \binom{k}{2} + 1$ (see Fig. 1 b).

Because CLIQUE is well known to be NP-complete [11] and our parameterized reductions are polynomial in k , it immediately follows that CUTTING AT MOST k VERTICES and CUTTING AT MOST k VERTICES WITH TERMINAL are NP-complete. \square

We observe that while we have an FPT-algorithm for CUTTING AT MOST k VERTICES when parameterized by k and t or by t only, it is unlikely that the

problem has a polynomial kernel (we refer to [7, 9, 16] for the formal definitions of kernels). Let G be a graph with s connected components G_1, \dots, G_s , and let $k \geq 1$, $t \geq 0$ be integers. Notice that (G, k, t) is a YES-instance of CUTTING AT MOST k VERTICES if and only if (G_i, k, t) is a YES-instance for some $i \in \{1, \dots, s\}$, because it always can be assumed that a solution is connected. By the results of Bodlaender et al. [3], this observation together with Theorem 11 immediately implies the following.

Theorem 12. *CUTTING AT MOST k VERTICES has no polynomial kernel when parameterized either by k and t or by t only, unless $\text{NP} \subseteq \text{coNP/poly}$.*

We will next show that when we consider the size of the separator t as the sole parameter, adding a terminal makes the problem harder. Indeed, while CUTTING AT MOST k VERTICES WITH TERMINAL with parameter t is trivially in XP, we next show that it is also W[1]-hard, in contrast to Theorem 5.

Theorem 13. *CUTTING AT MOST k VERTICES WITH TERMINAL is W[1]-hard with parameter t .*

Proof. Again, we prove the claim by a reduction from CLIQUE. Let (G, k) be a clique instance, $n = |V(G)|$ and $m = |E(G)|$; we create an instance (G', k', t, s) of CUTTING AT MOST k VERTICES WITH TERMINAL. The graph G' is constructed as follows. Create a new vertex s as the terminal. For each vertex and edge of G , add a corresponding vertex to G' , and add an edge between vertices v and e in G' when e is incident to v in G . Finally, connect all vertices of G' corresponding to vertices of G to the terminal s , and set $k' = n - k + m - \binom{k}{2} + 1$ and $t = k$. The construction is shown in Fig. 1 c).

If G has a k -clique K , then cutting away the k vertices of G' corresponding to K leaves exactly $n - k + m - \binom{k}{2} + 1$ vertices in the connected component of $G' - K$ containing s . Now suppose that $X \subseteq V(G')$ is a set with $s \in X$ such that $|X| \leq k'$ and $|N_{G'}(X)| \leq t$, and let $S = N_{G'}(X)$. Note that the elements of $V(G')$ that do not belong to X are exactly the elements $v \in S$ and the vertices corresponding to $e = uv$ such that $u, v \in S$ and $e \notin S$; denote this latter set of elements by E_0 . Since X is a solution, we have $|S| + |E_0| \geq \binom{k}{2} + k$, and thus $|E_0| \geq \binom{k}{2}$. But this is only possible if S is a k -clique in G . \square

Finally, we show that CUTTING k VERTICES BY EDGE-CUT WITH TERMINAL is also NP-hard, though the reduction we give is not sufficient to establish W[1]-hardness for k or t . Again, the problem is trivially in NP and in XP when parameterized by k or by t .

Theorem 14. *CUTTING k VERTICES BY EDGE-CUT WITH TERMINAL is NP-complete.*

Proof. We give a reduction from the CLIQUE problem. It is known that this problem is NP-complete for regular graphs [11]. Let (G, k) be an instance of CLIQUE, with G being a d -regular n -vertex graph. We create an instance (G', k', t, s) of CUTTING k VERTICES BY EDGE-CUT WITH TERMINAL as follows. The graph G' is constructed by starting from a base clique of size dn . One vertex in this base clique is selected as the terminal s , and we additionally distinguish d special vertices. For each $v \in V(G)$, we add a new vertex to G' , and add an edge between this vertex

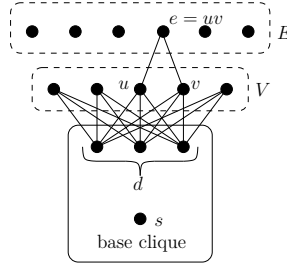


Figure 2: Construction of G' in the proof of Theorem 14.

and all of the d distinguished vertices of the base clique. For each edge $e = uv$ in G , we also add a new vertex to G' , and add edges between this vertex and vertices corresponding to u and v . The construction is shown in Fig. 2. We set $k' = dn + k + \binom{k}{2}$ and $t = dn - 2\binom{k}{2}$.

If G has a k -clique K , then selecting as X the base clique and all vertices of G' corresponding to vertices and edges of K gives a solution to (G', k', t, s) , as we have $|X| = dn + k + \binom{k}{2}$ and $|\partial(X)| = (dn - dk) + (dk - 2\binom{k}{2}) = dn - 2\binom{k}{2}$. For the other direction, consider any solution X to instance (G', k', t, s) . The set X clearly must contain the whole base clique. Let $V_0 \subseteq V$ and $E_0 \subseteq E$ be the subsets of X corresponding to vertices and edges of G , respectively. If $E_0 = \emptyset$, then $|\partial(X)| = dn$. Assume now that V_0 is fixed, and consider how adding vertices to E_0 changes $|\partial(X)|$. For each edge $e \in E(G)$, if neither of the endpoints of e is in V_0 , then adding e to E_0 adds 2 to $|\partial(X)|$. If exactly one of the endpoints of e is in V_0 , then adding e to E_0 does not change $|\partial(X)|$. Finally, if both of the endpoints of e are in V_0 , then adding e to E_0 reduces $|\partial(X)|$ by 2. Thus, in order to have $|\partial(X)| \leq dn - 2\binom{k}{2}$, we must have that $|E_0| \geq \binom{k}{2}$ and the endpoints of all edges in E_0 are in V_0 . But due to the requirement that $|X| \leq dn + k + \binom{k}{2}$, this is only possible if V_0 induces a clique in G . \square

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